Suggested Solution to Exercise 6

1. Find the tangent hyperplane passing the given point P on each of the graphs:

   (a)
   $$z = x^2 - y^2; \quad P(2, -3, -5).$$
   (b)
   $$y = z - \log \frac{x}{z} , \quad P(1, 1, 1) ,$$
   (c)
   $$w = \sin(x^2 + \pi z); \quad P(0, 1, 1, 0) .$$

   Solution.

   (a) z is a function of x and y. Its gradient is $\nabla z = (2x, -2y)$. The normal vector is given by $(-2x, 2y, 1)$. At $(2, -3, -5)$ it is given by $(-4, -6, 1)$. The tangent hyperplane at $(2, -3, -5)$ is
   $$(-4, -6, 1) \cdot ((x, y, z) - (2, -3, -5)) = 0 ,$$
   i.e.$-4x - 6y + z = 5$.

   (b) y is a function of x and z. Its gradient is given by $\nabla y = (-1/x, 1 + 1/z)$. The normal vector is given by $(1/x, 1, -1 - 1/z)$. At $(1, 1, 1)$ it is given by $(1, 1, -2)$. The tangent hyperplane at $(1, 1, 1)$ is
   $$(1, 1, -2) \cdot ((x, y, z) - (1, 1, 1)) = 0,$$
   i.e. $x + y - 2z = 0$.

   (c) w is a function of x, y and z. Its gradient is given by $\nabla w = (2x \cos(x^2 + \pi z), 0, \pi \cos(x^2 + \pi z)))$, so $\nabla w(0, 1, 1) = (0, 0, -\pi)$. The normal vector is $(-\nabla w, 1) = (0, 0, \pi, 1)$. The tangent hyperplane at $(0, 1, 1, 0)$ is
   $$(0, 0, \pi, 1) \cdot ((x, y, z, w) - (0, 1, 1, 0)) = 0 ,$$
   that is, $\pi z + w = \pi$.

2. Find the tangent plane and the normal line of each of the surfaces at the given point:

   (a)
   $$xy^2 - yz^2 + 6xyz = 6 , \quad P(1, 1, 1) .$$
   (b)
   $$x^2yz - e^{xy+1} = -2 , \quad P(1, -1, 1) .$$

   You should verify that it is a surface near the given point first.

   Solution.

   (a) Let $f(x, y, z) = xy^2 - yz^2 + 6xyz$. Then $\nabla f = (y^2 + 6yz, 2xy - z^2 + 6xz, -2yz + 6xy)$ and $\nabla f(1, 1, 1) = (7, 7, 4) \neq (0, 0, 0)$. By Theorem 6.2 $f = 6$ defines a surface near $(1, 1, 1)$. The tangent plane at $(1, 1, 1)$ is given by
   $$(7, 7, 4) \cdot ((x, y, z) - (1, 1, 1)) = 0,$$
   that is, $7x + 7y + 4z = 18$. The normal line at $(1, 1, 1)$ is given by
   $$(1, 1, 1) + t(7, 7, 4) , \quad t \in \mathbb{R} .$$

(b) Write $g(x, y, z) = x^2yz - e^{xy+1}$. Then

$$\nabla g = (2xyz - ye^{xy+1}, x^2z - xe^{xy+1}, x^2y) \ .$$

We have $\nabla g(1, -1, 1) = (-1, 0, -1) \neq (0, 0, 0)$. Hence $g = -2$ defines a surface near $(1, -1, 1)$. The tangent plane at $(1, -1, 1)$ is given by

$$(-1, 0, -1) \cdot ((x, y, z) - (1, -1, 1)) = 0 \ ,$$

or $x + z = 2$. The normal line at $(1, -1, 1)$ is given by

$$(1, -1, 1) + t(-1, 0, -1) \ , \quad t \in \mathbb{R} \ .$$

3. Use implicit differentiation to find

(a) y' and y'' for $x^2 + 2xy - y^2 = a^2$.

(b) y' and y'' for $y - \delta \sin y = x, \quad \delta \in (0, 1)$.

The solutions are allowed to depend on y.

Solution.

(a) Differentiate both sides with respect to x to $x^2 + 2xy - y^2 = a^2$ yields

$$2x + 2y + 2xy' - 2yy' = 0,$$

so

$$y' = \frac{x + y}{y - x} \ .$$

One more differentiation gives

$$2 + 2y' + 2y' + 2xy'' - 2(y')^2 - 2yy'' = 0,$$

that is,

$$y'' = \frac{1 + 2y' - y'^2}{y - x}.$$

You may plug in the expression of y' so that the right hand side contains x and y only, but this is optional.

(b) Differentiate both sides with respect to x to $y - \delta \sin y = x$ yields

$$y' - y'\delta \cos y = 1,$$

which gives

$$y' = \frac{1}{1 - \delta \cos y} \ .$$

One more differentiation gives

$$y'' - y''\delta \cos y + (y')^2 \delta \sin y = 0,$$

that is,

$$y'' = \frac{\delta {y'}^2 \sin y}{\delta \cos y - 1}.$$

4. Use implicit differentiation to find the first and second partial derivatives of $z = z(x, y)$:

(a)
$$x + y + z = e^z \ ,$$

(b)
$$\sin(x + y) - 6\cos(y + z) = x \ .$$

Solution.

(a) First we get $1 + z_x = z_x e^z$, or
$$z_x = \frac{1}{e^z - 1}.$$
Then $z_{xx} = z_{xx}e^z + (z_x)^2 e^z$ which gives
$$z_{xx} = \frac{(z_x)^2 e^z}{1 - e^z} \ .$$
Similarly we get
$$z_y = \frac{1}{e^z - 1},$$
and
$$z_{yy} = \frac{(z_y)^2 e^z}{1 - e^z} \ .$$
Finally, differentiate both sides with respect to y to $1 + z_x = z_x e^z$ yields
$$z_{xy} = \frac{z_x z_y e^z}{1 - e^z} \ .$$

(b)
$$z_x = \frac{1 - \cos(x + y)}{6\sin(y + z)} \ .$$
$$z_{xx} = \frac{\sin(x + y) - 6\cos(y + z)z_x^2}{6\sin(y + z)} \ .$$
$$z_y = -\frac{\cos(x + y) + 6\sin(y + z)}{6\sin(y + z)} \ .$$
$$z_{yy} = \frac{\sin(x + y) - 6\cos(y + z)(1 + z_y)^2}{6\sin(y + z)} \ .$$
$$z_{xy} = \frac{\sin(x + y) - 6\cos(y + z)z_x(1 + z_y)}{6\sin(y + z)} \ .$$

5. Find all first and second partial derivatives of $y = y(x, z)$ satisfying
$$x^2 y - 6y^2 z + xz^2 = 8 \ ,$$
at $(1, 1, -1)$.

Solution. Differentiate both sides with respect to x to $x^2y - 6y^2z + xz^2 = 8$ yields
$$2xy + x^2 y_x - 12yy_x z + z^2 = 0 \ .$$
Therefore, at the point $(1, 1, -1)$,
$$y_x = \frac{2xy + z^2}{12yz - x^2} = -\frac{3}{13}.$$

Similarly,

$$y_z = \frac{-6y^2 + 2xz}{12yz - x^2} = \frac{8}{13},$$

and

$$y_{xx} = -\frac{2y + 4xy_x - 12(y_x)^2 z}{x^2 - 12yz} = -\frac{368}{2197} \ ,$$

$$y_{zz} = -\frac{12(y_z)^2 z - 12yy_z + 2x}{12yz - x^2} = \frac{1390}{2197},$$

$$y_{xz} = -\frac{2xy_z - 12y_z y_x z - 12yy_x + 2z}{12yz - x^2} = \frac{213}{2197} \ .$$

(Hope the numbers are correct.)

6. Find the condition that z can be viewed as a function of x, y in the relation $F(xz, yz) = 0$. Then find z_x and z_{xx}.

Solution. Let $G(x, y, z) = F(u, v) = F(xz, yz)$. By the chain rule,

$$G_z = xF_u(xz, yz) + yF_v(xz, yz) \ .$$

Now, by Implicit Function Theorem, z is a function of x, y in the relation $G(z) = 0$ if $G_z \neq 0$, i.e.

$$xF_u(xz, yz) + yF_v(xz, yz) \neq 0.$$

When this holds, differentiate both sides with respect to x to $F(xz, yz) = 0$ yields

$$F_u \cdot (z + xz_x) + F_v \cdot (yz_x) = 0 \ .$$

Therefore,

$$z_x = -\frac{zF_u(xz, yz)}{xF_u(xz, yz) + yF_v(xz, yz)}.$$

Differentiate both sides with respect to x to $F_u \cdot (z + xz_x) + F_v \cdot (yz_x) = 0$ yields

$$F_{uu} \cdot (z+xz_x)^2 + F_{uv} \cdot (z+xz_x)(yz_x) + F_u \cdot (2z_x + xz_{xx}) + F_{vu} \cdot (yz_x)(z+xz_x) + F_{vv} \cdot (yz_x)^2 + F_v \cdot (yz_{xx}) = 0 \ .$$

Therefore,

$$z_{xx} = -\frac{F_{uu} \cdot (z + xz_x)^2 + F_{uv} \cdot (z + xz_x)(yz_x) + 2F_u \cdot z_x + F_{vu} \cdot (yz_x)(z + xz_x) + F_{vv} \cdot (yz_x)^2}{xF_u + yF_v}.$$

7. Let Φ be a function defined on the intersection of the zero set of two functions

$$g(x, y, z) = 0, \quad h(x, y, z) = 0 \ .$$

Write down the condition that the intersection can be parametrized by x. Then find $\dfrac{d\Phi}{dx}$ and $\dfrac{d^2\Phi}{dx^2}$.

Solution. By Implicit Function Theorem, the intersection is a curve parametrized by x if

$$\begin{vmatrix} g_y & g_z \\ h_y & h_z \end{vmatrix} \neq 0 \ .$$

If so, letting $(x, y(x), z(x))$ be the curve of intersection parametrized by x, we differentiate $g(x, y(x), z(x)) = 0$ and $h(x, y(x), z(x)) = 0$ to get

$$\begin{cases} g_x + g_y y_x + g_z z_x = 0, \\ h_x + h_y y_x + h_z z_x = 0 \ . \end{cases}$$

Hence (y_x, z_x) can be expressed in terms of first partial derivatives of g and h. Similarly, all the first and second partial derivatives of y and z can be expressed in terms of first and second partial derivatives of g and h. Therefore,

$$\frac{d}{dx}\Phi(x, y(x), z(x)) = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} y_x + \frac{\partial \Phi}{\partial z} z_x$$

and

$$\begin{aligned} \frac{d^2}{dx^2}\Phi(x, y(x), z(x)) &= \Big(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y \partial x} y_x + \frac{\partial^2 \Phi}{\partial z \partial x} z_x\Big) \\ &+ \Big(\frac{\partial^2 \Phi}{\partial x \partial y} y_x + \frac{\partial^2 \Phi}{\partial y^2} y_x^2 + \frac{\partial^2 \Phi}{\partial z \partial y} y_x z_x + \frac{\partial \Phi}{\partial y} y_{xx}\Big) \\ &+ \Big(\frac{\partial^2 \Phi}{\partial x \partial z} z_x + \frac{\partial^2 \Phi}{\partial y \partial z} y_x z_x + \frac{\partial^2 \Phi}{\partial z^2} z_x^2 + \frac{\partial \Phi}{\partial z} z_{xx}\Big) \\ &= \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} y_x^2 + \frac{\partial^2 \Phi}{\partial z^2} z_x^2 \\ &+ 2\frac{\partial^2 \Phi}{\partial x \partial y} y_x + 2\frac{\partial^2 \Phi}{\partial x \partial z} z_x + 2\frac{\partial^2 \Phi}{\partial y \partial z} y_x z_x \\ &+ \frac{\partial \Phi}{\partial y} y_{xx} + \frac{\partial \Phi}{\partial z} z_{xx} \ . \end{aligned}$$

8. Explain why each of the following system defines a curve $\gamma(z) = (x(z), y(x), z)$ in $\mathbb{R}^3$ and then find the first derivatives of γ:

(a)
$$x + y + z = 0, \quad x + y^2 + z^4 = 1 \ ,$$

(b)
$$x^2 + y^2 = \frac{1}{2}z^2, \quad x + y + z = 2, \quad \text{at } (1, -1, 2) \ .$$

Solution.

(a) The Jacobian matrix associated to the functions $g(x, y, z) = x + y + z = 0$ and $h(x, y, z) = x + y^2 + z^4 = 0$ is given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2y & 4z^3 \end{bmatrix} .$$

We claim that this matrix has rank 2 at each $(x, y, z) \in \mathbb{R}^3$ satisfying the system. For, each $(x, y, z) \in \mathbb{R}^3$ satisfying the system, if $\begin{vmatrix} 1 & 1 \\ 1 & 2y \end{vmatrix} \neq 0$, then the matrix has rank 2; if $\begin{vmatrix} 1 & 1 \\ 1 & 2y \end{vmatrix} = 0$, then $y = 1/2$. Therefore, since (x, y, z) satisfies the system, we have

$$x + \frac{1}{2} + z = 0, \quad x + \frac{1}{4} + z^4 = 1 \ ,$$

which implies $z^4 - z = 5/4$. Now if $\begin{vmatrix} 1 & 1 \\ 1 & 4z^3 \end{vmatrix} = 0$, then $z = 4^{-1/3}$, and one checks that $4^{-4/3} - 4^{-1/3} \neq 5/4$. Therefore, $\begin{vmatrix} 1 & 1 \\ 1 & 4z^3 \end{vmatrix} \neq 0$ if $\begin{vmatrix} 1 & 1 \\ 1 & 2y \end{vmatrix} = 0$. As a result, this matrix is of rank 2 everywhere. By Theorem 6.5, the solution set always defines a curve everywhere. When $y \neq 1/2$, the curve is parametrized by $z : (x(z), y(z), z)$. Its tangent is $(x'(z), y'(z), 1)$, where (x', y') can be obtained by differentiating both sides of the two defining functions $g(x(z), y(z), z) = 0$ and $h(x(z), y(z), z) = 0$ with respect to z, that is, $x' + y' + 1 = 0$ and $x' + 2yy' + 4z^3 = 0$. We get

$$x' = \frac{-2y + 4z^3}{2y - 1}, \quad y' = \frac{-4z^3 + 1}{2y - 1} .$$

(b) The Jacobian matrix associated to the functions $g(x, y, z) = x^2 + y^2 - \frac{1}{2}z^2 = 0$ and $h(x, y, z) = x + y + z - 2 = 0$ is given by

$$\begin{bmatrix} 2x & 2y & -z \\ 1 & 1 & 1 \end{bmatrix} ,$$

which is equal to

$$\begin{bmatrix} 2 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

at $(1, -1, 2)$ Since $\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} \neq 0$, the matrix has rank 2 at P. By Theorem 6.5, the curve can be parametrized as $(x(z), y(z), z)$. Differentiating both sides with respect to z to $g(x(z), y(z), z) = 0$ and $h(x(z), y(z), z) = 0$ yields $2xx' + 2yy' - z = 0$ and $x' + y' + 1 = 0$. At $P(1, -1, 2)$, we have $2x' - 2y' - 2 = 0$ and $x' + y' + 1 = 0$. We have $x' = 0$ and $y' = -1$. The tangent vector at P is $(x', y'z') = (0, -1, 1)$ and the tangent line passing through P is given by

$$(1, -1, 2) + (0, -1, 1)t , \quad t \in \mathbb{R} .$$

Note. It cannot be parametrized in x.

9. * The spherical coordinates are given by

$$x = r\cos\theta\sin\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\varphi ,$$

where

$$r \geq 0, \quad \theta \in [0, 2\pi), \quad \varphi \in [0, \pi] .$$

(a) Give a geometric interpretation of this coordinates.

(b) Show that

$$r = \sqrt{x^2 + y^2 + z^2} , \quad \theta = \arctan\frac{y}{x} , \quad \text{and } \varphi = \arccos\frac{z}{r} .$$

(c) Express f_x and f_{xx} in terms of f_r, f_θ, and f_φ.

(d) * Show that the three dimensional Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 ,$$

in spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin^2\varphi}\frac{\partial^2 f}{\partial\theta^2}+\frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial f}{\partial\varphi}\right)=0\ .$$

Solution.

(a) For $r=0$, (x,y,z) is the origin.
For $r>0$, (x,y,z) can be regarded as a point on the sphere in $\mathbb{R}^3$ with radius r. φ and θ can be regarded the "latitude" and the "longitude" of (x,y,z) on the sphere respectively.

(b) (i)

$$\begin{aligned}x^2+y^2+z^2&=(r\cos\theta\sin\varphi)^2+(r\sin\theta\sin\varphi)^2+(r\cos\varphi)^2\\&=(r\sin\varphi)^2(\cos^2\theta+\sin^2\theta)+(r\cos\varphi)^2\\&=(r\sin\varphi)^2+(r\cos\varphi)^2\\&=r^2\ .\end{aligned}$$

Therefore,

$$r=\sqrt{x^2+y^2+z^2}\ .$$

(ii) Dividing the first equation of the definition of spherical coordinates by the second equation yields

$$\frac{y}{x}=\tan\theta\ .$$

Therefore,

$$\theta=\arctan\frac{y}{x}\ .$$

(iii) Directly consider the third equation of the definition of spherical coordinates, we have

$$\varphi=\arccos\frac{z}{r}\ .$$

(c) We omit this lengthy but straightforward computation.

10. * Let

$$x=t+\frac{1}{t},\quad y=t^2+\frac{1}{t^2},\quad z=t^3+\frac{1}{t^3}\ .$$

Find y_x, z_x, y_{xx} and z_{xx}.

Solution. Note that $\dfrac{dx}{dt}=1-\dfrac{1}{t^2}$. Therefore, when $t\neq\pm1$, $\dfrac{dx}{dt}\neq 0$, and hence by Implicit Function Theorem t is a function of x, say $t=g(x)$ with

$$g'(x)=\frac{1}{1-\dfrac{1}{t^2}}=\frac{t^2}{t^2-1}.$$

Now we can regard $y=y(g(x))$ and $z=z(g(x))$. Differentiate both sides with respect to x to above equations yields

$$y_x=y_tg'(x)=\left(2t-\frac{2}{t^3}\right)\left(\frac{t^2}{t^2-1}\right)=2x\ ,$$

and

$$z_x = z_t g'(x) = \left(3t^2 - \frac{3}{t^4}\right)\left(\frac{t^2}{t^2-1}\right) = 3(y+1) \ .$$

Differentiate the above equations with respect to x again yield

$$y_{xx} = y_{tt}(g'(x))^2 + y_t g''(x) = 2,$$

and

$$z_{xx} = z_{tt}(g'(x))^2 + z_t g''(x) = 6x \ .$$

11. * Let

$$x = u\cos\frac{v}{u}, \quad y = u\sin\frac{v}{u} \ .$$

Find u_x, u_y, v_x, v_y. Justify the inverse function exists first.

Solution. The Jacobian matrix of x, y with respect to u, v is given by

$$\begin{bmatrix} \cos\frac{v}{u} + \frac{v}{u}\sin\frac{v}{u} & -\sin\frac{v}{u} \\ \sin\frac{v}{u} - \frac{v}{u}\cos\frac{v}{u} & \cos\frac{v}{u} \end{bmatrix} \ .$$

The determinant of Jacobian matrix is given by

$$\begin{vmatrix} \cos\frac{v}{u} + \frac{v}{u}\sin\frac{v}{u} & -\sin\frac{v}{u} \\ \sin\frac{v}{u} - \frac{v}{u}\cos\frac{v}{u} & \cos\frac{v}{u} \end{vmatrix} = 1 \neq 0$$

Therefore, by Inverse Function Theorem, the inverse function exists. The Jacobians of the given map and its inverse is related by

$$\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ .$$

Therefore,

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}^{-1} = \begin{bmatrix} \cos\frac{v}{u} & \sin\frac{v}{u} \\ -\sin\frac{v}{u} + \frac{v}{u}\cos\frac{v}{u} & \cos\frac{v}{u} + \frac{v}{u}\sin\frac{v}{u} \end{bmatrix} \ .$$